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Solution of heat transfer problems for stochastic boundary conditions

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Abstract

Solutions are known for a great deal of heat transfer problems with stationary or analytically given transient boundary conditions. In this paper the well-known *Duhamel theorem* will be extended for boundary conditions not stated analytically. The fact that the energy supply to a system depends on time in a stochastic manner is a problem in many technical applications. It will be shown by comparison with well-known analytical calculations, that the extended method of Duhamel can be used for calculating the temperature distributions and heat fluxes in any case of transient variations of heat supply or temperatures forced upon the system. The method leads to a long series of error or trigonometrical functions. In order to compute these in real time, an EDP with high capacity and velocity is necessary. But that should be no problem nowadays.

Keywords: Duhamel theorem; Heat conduction in solids; Stochastic boundary conditions; Approximate solutions

1. Introduction

The Fourier's differential equation describes transient and local temperatures in a solid flat—plates, cylinders, spheres—with energy supply on their surfaces or by heat sources. For one-dimensional heat transfer in a plate without heat sources, the differential equation is

$$\frac{1}{a}\frac{\partial \vartheta}{\partial t} = \frac{\partial^2 \vartheta}{\partial x^2} \tag{1}$$

For many cases of boundary conditions, solutions are known [1–3] in the general form

$$\vartheta = \mathrm{f}\left(\frac{at}{x^2}, \frac{\alpha s}{\lambda}, \frac{x}{s}\right) \tag{2}$$

Thereby the boundary conditions are given as analytic functions: Sudden transition (jump), linear, periodic, or similar temperature rises in the surrounding medium or on the surface of the solid.

Due to technical problems, it is often impossible to express the boundary conditons in an analytical form, because they are stochastic. This is, for instance, the case when in a combustion engine the efficiency or when on a surface the solar radiations by more or less clouding are being permanently altered. The temperatures in the engine or on the surface are being changed repeatedly, Fig. 1.

These permanent variations can be described using the well-known *Duhamel theorem* [4], which is also used in the field of control engineering as convolution integral. An explanation of this method is given in [2].

Examples in the literature describe the use of the theorem for analytically given transient boundary conditions only [1,2,5]. Extending it to stochastic variations in heat transfer processes is yet unknown but an obvious thing to do.

The problems mentioned above can be solved by numerical integration of Eq. (1), too, in which with every alteration

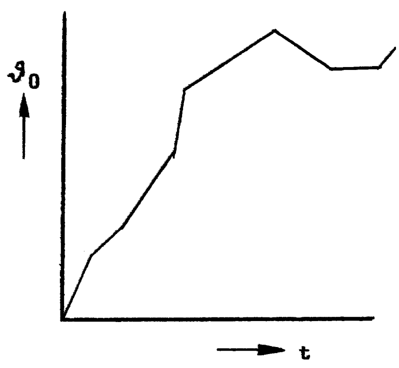

Fig. 1. Surface temperatures by stochastic alterations of the energy supply to a solid.

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Nomenclature

A	amplitude	K
a	thermal diffusivity	$m^2 \cdot s^{-1}$
b	heat penetration coefficient	$W \cdot s^{1/2} \cdot m^{-2} \cdot K^{-1}$
c	heat capacity	$J \cdot kg^{-1} \cdot K^{-1}$
g	gradient of temperature	$K \cdot s^{-1}$
k	overall coefficient of heat transfer	$W \cdot m^{-2} \cdot K^{-1}$
N	total number of steps	
n	step variable	
$\dot{q}$	heat flux	$W \cdot m^{-2}$
s	thickness	m
T	total time	s
t	time variable	s
x	coordinate	m
z	dimensionless parameter, Eqs. (25), (68)	

Greek symbols

α	heat transfer coefficent	$W \cdot m^{-2} \cdot K^{-1}$
ϑ	temperature	K
τ	time, integration variable	s
λ	heat conductivity	$W \cdot m^{-1} \cdot K^{-1}$
μ_m	roots of transcendental equations	
ρ	density	$kg \cdot m^{-3}$
ξ	dimensionless parameter, Eq. (9)	
ω	frequency	s^{-1}

Subscripts

O	surface
M	medium
m	mean value
0	for $x = 0$
s	for $x = s$

in the boundary conditions the temperature field at this time is the initial condition for the next integration step. There is just one disadvantage with this method—as with all numerical calculations—there is no information about the effects of the physical parameters in the investigated process. When solving it with Duhamel theorem, the effects of the parameters remain recognizable.

2. Handling and testing the Duhamel theorem

The theorem of Duhamel is defined by the following [1]: If the temperature at the time t in a solid in which the initial temperature is zero, while its surface is kept at a steady temperature, is defined as follows

$$\vartheta = \mathrm{F}(x, y, z, t) \tag{3}$$

then if the temperature on the surface is a transient function

$$\vartheta_O = \phi(t) \tag{4}$$

the temperature in the solid is to be calculated by the integral

$$\vartheta = \int_0^t \phi(\tau) \cdot \frac{\partial}{\partial t} \mathrm{F}(x, y, z, t - \tau)\, \mathrm{d}\tau \tag{5}$$

The solution for one-dimensional heat conduction in a semi-infinite solid follows from Eq. (1)

$$\vartheta = f\left(\operatorname{erfc} \frac{x}{2\sqrt{at}}\right) \tag{6}$$

and therefore for the function F in Eq. (5)

$$\mathrm{F}(x, t - \tau) = \operatorname{erfc} \frac{x}{2\sqrt{a(t-\tau)}} \tag{7}$$

For a linear temperature rise on the surface, for example, Eq. (4) changes to

$$\vartheta_O = g \cdot t \quad \text{with } g = \Delta\vartheta_O / \Delta t \tag{8}$$

The integration of Eq. (5) with Eq. (4), respectively Eq. (8) results in

$$\vartheta = g \cdot t \left\{ \left(1 + 2\xi^2\right) \operatorname{erfc}(\xi) - \frac{2}{\sqrt{\pi}} \xi e^{-\xi^2} \right\} = 4 \cdot g \cdot t \left(i^2 \operatorname{erfc}(\xi)\right) \tag{9}$$

Here $\xi = \frac{x}{2\sqrt{at}}$, the complementary error function $\operatorname{erfc}(\xi) = 1 - \operatorname{erf}(\xi)$ and the repeated integrals

$$i^n \operatorname{erfc}(\xi) = \int_\xi^\infty i^{n-1} \cdot \operatorname{erfc}(\xi)\, \mathrm{d}\xi \tag{10}$$

For the heat flux in the solid, the following equation applies

$$\dot{q} = -\lambda \frac{\partial \vartheta}{\partial x} = \frac{2b\Delta\vartheta_O}{\sqrt{t}} \left[\frac{1}{\sqrt{\pi}} e^{-\xi^2} - \xi \operatorname{erfc}(\xi) \right] = \frac{2b\Delta\vartheta_O}{\sqrt{t}} \cdot i^1 \operatorname{erfc}(\xi) \tag{11}$$

with $b = \sqrt{\lambda c \rho}$. This example describes an analytically given linear temperature rise in the time. Regarding the problems mentioned above with stochastic alterations of the temperature—rise or decrease, the important application of Duhamel's theorem concerns the fact, that many alterations of temperatures or heat fluxes can successively happen any time.

Regarding the above example with a linear temperature rise, this statement means that the continuous rise can be substituted by a series of N temperature steps, Fig. 2. Eq. (8) will be reproduced with a sufficient number of small steps. This will be shown in the following to explain the application of Duhamel's theorem. In the case of only one temperature step $\Delta\vartheta_0$ ($N = 1$) Eq. (6) accordingly applies:

$$\vartheta = \Delta\vartheta_O \operatorname{erfc}(\xi) \tag{12}$$

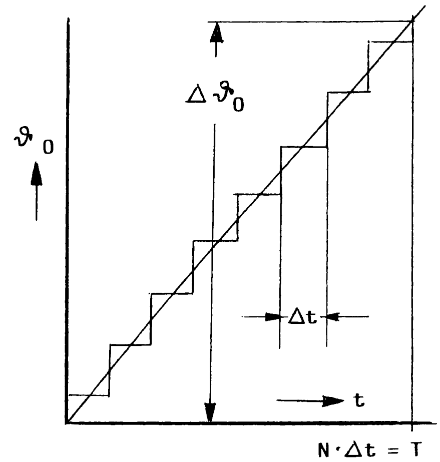

Fig. 2. Linear temperature rise, divided in N jumps.

For subdividing in two steps $\Delta\vartheta_O/2$ ($N=2$), while the second step follows after $T/2$:

$$\vartheta = \frac{\Delta\vartheta_O}{2}\operatorname{erfc}\frac{x}{2\sqrt{aT}} + \frac{\Delta\vartheta_O}{2}\operatorname{erfc}\frac{x}{2\sqrt{a(T-T/2)}}$$

$$= \frac{\Delta\vartheta_O}{2}\left[\operatorname{erfc}\frac{x}{2\sqrt{aT}} + \operatorname{erfc}\frac{x}{2\sqrt{aT/2}}\right]$$

$$= \frac{\Delta\vartheta_O}{2}\left[\operatorname{erfc}(\xi) + \operatorname{erfc}\left(\xi\cdot\sqrt{2}\right)\right] \tag{13}$$

Accordingly for three steps $\Delta\vartheta_O/3$ ($N=3$):

$$\vartheta = \frac{\Delta\vartheta_O}{3}\left[\operatorname{erfc}\frac{x}{2\sqrt{aT}} + \operatorname{erfc}\frac{x}{2\sqrt{a(T-T/3)}} + \operatorname{erfc}\frac{x}{2\sqrt{a(T-2T/3)}}\right]$$

$$= \frac{\Delta\vartheta_O}{3}\left[\operatorname{erfc}(\xi) + \operatorname{erfc}\left(\xi\cdot\sqrt{\frac{3}{2}}\right) + \operatorname{erfc}\left(\xi\cdot\sqrt{3}\right)\right] \tag{14}$$

Generally, for N steps with a temperature rise $\Delta\vartheta_O/N$ in every step and with $T = N\cdot\Delta t$:

$$\vartheta = \frac{\Delta\vartheta_O}{N}\cdot\sum_{n=1}^{N}\operatorname{erfc}\left(\xi\sqrt{N/n}\right) \tag{15}$$

By increasing N, finally $N\to\infty$, Eqs. (9) and (15) must be identical, i.e.,

$$4\cdot i^2\operatorname{erfc}(\xi) = \lim_{N\to\infty}\left[\frac{1}{N}\sum_{n=1}^{N}\operatorname{erfc}\left(\xi\cdot\sqrt{N/n}\right)\right] \tag{16}$$

This statement can be confirmed by checking. The question for a practical realization is how many steps are necessary to describe a given function (i.e., the linear rise) with sufficient accuracy. In Table 1 the series issue from Eq. (15) is calculated for different numbers of N. It can be seen that a relative great number is needed to reach a complete conformity with Eq. (9), but for technical calculations approx. $N = 40$ may be sufficient.

Table 1
Linear temperature rise, Eq. (15) as function of N, ξ

ϑ/gt	$\xi = 0.1$	0.5	1.0	2.0
$N=1$	0.8875	0.4795	0.1573	0.00468
5	0.8384	0.3295	0.0732	0.00130
10	0.8235	0.3041	0.0648	0.00101
20	0.8121	0.2919	0.0608	0.00088
40	0.8041	0.2859	0.0588	0.00082
∞	0.7936 $= 4\cdot i^2\operatorname{erfc}(\xi)$	0.2799	0.0568	0.00076

Table 2
Heat flux at $x = 0$ for a linear temperature rise, Eq. (17) as function of N, ξ

$\frac{\dot q\cdot\sqrt{T}}{b\Delta\vartheta_O}$	$\xi = 0.1$	0.5	1.0	2.0
$N=1$	0.5586	0.4394	0.2076	0.01033
5	0.7936	0.4440	0.1215	0.00311
10	0.8614	0.4220	0.1110	0.00250
20	0.9066	0.4102	0.1057	0.00222
40	0.9317	0.4048	0.1031	0.00209
∞	0.9396 $= 2\cdot i^1\operatorname{erfc}(\xi)$	0.3993	0.1005	0.00202

The heat flux $\dot q$ for a linear temperature rise is to be calculated from Eq. (15) by differentiation:

$$\dot q = -\lambda\frac{\partial\vartheta}{\partial x} = \frac{b\Delta\vartheta_O}{\sqrt{\pi}\cdot\sqrt{T}}\frac{1}{N}\sum_{n=1}^{N}\sqrt{N/n}\cdot e^{-\xi^2 N/n} \tag{17}$$

This equation must be identical for small steps in the time (i.e., great numbers of N) with Eq. (11). This can be seen in Table 2. It can be shown for other functions for the temperature on the surface $\vartheta_O(t)$, e.g., for a periodical sine or cosine function, that the method of Duhamel reproduces it arbitrarily exact according to the number of steps, see Appendix A.

3. Generalization of Duhamel's theorem

3.1. Sudden transition (jump) of the temperature by a boundary condition of the 1. kind

Duhamel's theorem describes in general the effects of variable boundary conditions on the temperature field. In the foregoing text, this was shown for a linear temperature rise on the surface. It is appropriate with non-linear variations to hold the step of time Δt constant and to insert the height of the temperature jumps $\Delta\vartheta_O$ according to the given function, Fig. 3.

The temperature distribution $\vartheta = \mathrm{f}(x,t)$ is therefore described by the following equation at the time $T = N\cdot\Delta t$

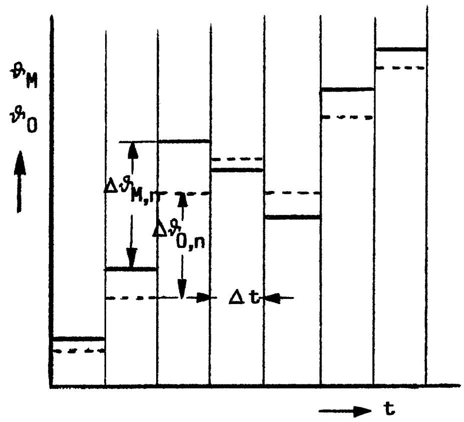

Fig. 3. Jumps of medium and surface temperatures by stochastic alterations.

$$\begin{aligned}\vartheta &= \Delta\vartheta_{O,1}\operatorname{erfc}\frac{x}{2\sqrt{aN\Delta t}} + \Delta\vartheta_{O,2}\operatorname{erfc}\frac{x}{2\sqrt{a(N-1)\Delta t}} \\ &\quad + \Delta\vartheta_{O,3}\operatorname{erfc}\frac{x}{2\sqrt{a(N-2)\Delta t}} + \cdots \\ &\quad + \Delta\vartheta_{O,N-1}\operatorname{erfc}\frac{x}{2\sqrt{a\cdot 2\Delta t}} + \Delta\vartheta_{O,N}\operatorname{erfc}\frac{x}{2\sqrt{a\Delta t}} \\ &= \sum_{n=1}^{N}\Delta\vartheta_{O,n}\operatorname{erfc}\frac{x}{2\sqrt{a(N+1-n)\Delta t}} \end{aligned} \tag{18}$$

with $\Delta\vartheta_{O,n} = \vartheta_{O,n} - \vartheta_{O,n-1}$. The size of the time steps Δt and therefore the number of steps N will be chosen by means of the accuracy required.

The heat flux $\dot{q}$ is derived by differentiation of Eq. (18)

$$\dot{q} = \frac{b}{\sqrt{\pi}\cdot\sqrt{T}}\sum_{n=1}^{N}\Delta\vartheta_{O,n}\sqrt{\frac{N}{N+1-n}} \times e^{-(x/2\sqrt{a(N+1-n)\Delta t})^2} \tag{19}$$

It is possible to calculate the resulting temperature distribution and the heat flux with the aid of Eqs. (18) and (19) for every additional change in the temperature or in the boundary conditions.

The series $\sum n$ will be extended by one term with every step. So the series can get very long. It can, however, be observed that the first terms $\vartheta_{O,1}, \vartheta_{O,2}, \ldots$ get successively smaller with the time. For long series it is therefore justified to replace the first terms by an evaluation (see Section 5). It has to be remarked furthermore that every additional step requires a new calculation of the whole series, because the number of steps N is not only an indicator for the number of steps but is a parameter by itself.

3.2. Sudden temperature changes by boundary conditions of the 3. kind

When solving technical problems, changes due to heat transfer from a fluid to a solid—the 3. kind of boundary conditions—are more important than the boundary conditions of the 1. kind. The following conditions have therefore to be complied with: The heat flux by heat transfer from the fluid on the solid must be identical with the heat conduction from the surface $x = 0$ into the solid, see Fig. 4

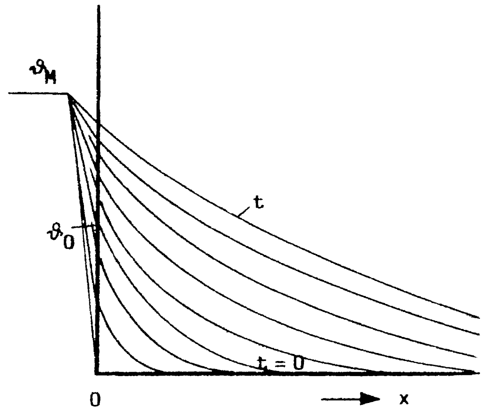

Fig. 4. Temperature distribution in a semi-infinite solid by a boundary condition of the 3. kind.

$$\alpha\big(\vartheta_M(t) - \vartheta_O(t)\big) = \dot{q}|_{x=0} \tag{20}$$

The heat flux on the surface for a sudden change is given by Eq. (19) for $x = 0$

$$\dot{q}|_{x=0} = \frac{b}{\sqrt{\pi}\cdot\sqrt{T}}\sum_{n=1}^{N}\Delta\vartheta_{O,n}\frac{\sqrt{N}}{\sqrt{N+1-n}} \tag{21}$$

$$= \frac{b}{\sqrt{\pi}\cdot\sqrt{\Delta t}}\sum_{n=1}^{N}\frac{\Delta\vartheta_{O,n}}{\sqrt{N+1-n}} \tag{22}$$

$$= \frac{b}{\sqrt{\pi}\cdot\sqrt{\Delta t}}\sum_{n=0}^{N-1}\frac{\Delta\vartheta_{O,N-n}}{\sqrt{n+1}} \tag{23}$$

Eqs. (22) and (23) differ from one another in the numbering of the terms N: In Eq. (22) $n = 1$ is the first step, and in Eq. (23) $n = 0$ is the last term N of the series.

The temperatures at the time T are the sums of the jumps

$$\vartheta_M(T) = \sum_{n=1}^{N}\Delta\vartheta_{M,n}, \qquad \vartheta_O(T) = \sum_{n=1}^{N}\Delta\vartheta_{O,n} \tag{24}$$

The temperatures on the surface ϑ_O are at first unknown and must be calculated with the aid of Eq. (20) (or by iteration, if further influences dependent on the surface temperature are given, e.g., radiation, condensation, heat tranfer by free convection [7]).

The physical assumption for the connection of Eqs. (20)–(24) is an immediate reaction of the surface temperature to a change in the fluid temperature. The error by this assumption will disappear, if the steps Δt are choosen as small as possible.

By elimination of the unknown surface temperature the heat flux will be determined by the equations [1]: for just a single jump of the fluid temperature $\Delta\vartheta_M$ ($N = n = 1$)

$$\vartheta_M - \vartheta_O = (\vartheta_M - \vartheta_{O,0})e^{z^2}\operatorname{erfc}(z) \tag{25}$$

and for repeated jumps $\Delta\vartheta_{M,n}$

$$\vartheta_{M,N} - \vartheta_{O,N} = \sum_{n=1}^{N} \Delta\vartheta_{M,n} \cdot e^{z^2} \operatorname{erfc}(z) \tag{26}$$

with $z = \frac{\alpha}{b}\sqrt{\Delta t(N+1-n)}$, or

$$\vartheta_{M,N} - \vartheta_{O,N} = \sum_{n=0}^{N-1} \Delta\vartheta_{M,N-n} \cdot e^{z'^2} \operatorname{erfc}(z') \tag{27}$$

with $z' = \frac{\alpha}{b}\sqrt{\Delta t(n+1)}$.

Using Eqs. (26) or (27), the heat flux $\dot{q}|_{x=0} = \alpha(\vartheta_{M,N} - \vartheta_{O,N})$ is determined directly. For the calculation of the terms $e^{z^2} \cdot \operatorname{erfc}(z)$, an integration within the series is, indeed, necessary. The approximation equation (28) holds for $z > 3$

$$e^{z^2} \cdot \operatorname{erfc}(z) \approx \frac{1}{\sqrt{\pi}}\left(\frac{1}{z} - \frac{1}{2z^3} + \frac{1\cdot 3}{2^2 z^5}\right) \tag{28}$$

3.3. Jumps of the heat flux, boundary conditions of the 2. kind

A heat flux is generated with a gradient $\frac{\partial\vartheta}{\partial x}$

$$\dot{q} = -\lambda\frac{\partial\vartheta}{\partial x} \tag{29}$$

This heat flux satisfies the same differential equation as that of the temperature (Fourier's equation)

$$\frac{\partial^2\dot{q}}{\partial x^2} = \frac{1}{a}\frac{\partial\dot{q}}{\partial t} \tag{30}$$

Therefore the solutions of Eq. (30) are in accordance to those of the temperature. For constant heat flux on the surface of a semi-infinite solid with sudden addition of $\dot{q}|_{x=0}$ the following is valid

$$\dot{q} = \dot{q}|_{x=0} \cdot \operatorname{erfc}\frac{x}{2\sqrt{at}} \tag{31}$$

The temperature distribution in the solid by this heat flux is obtained by integration of Eq. (29)

$$\vartheta - \vartheta_{O,0} = \frac{2}{b}\sqrt{t} \cdot \dot{q}|_{x=0} \cdot i^1 \operatorname{erfc}\frac{x}{2\sqrt{at}} \tag{32}$$

and for the surface temperature

$$\vartheta_O - \vartheta_{O,0} = \frac{2}{\sqrt{\pi}\,b}\sqrt{t} \cdot \dot{q}|_{x=0} \tag{33}$$

It is a generally known fact that the temperature gets $\sqrt{t}$ by $\dot{q}|_{x=0} = \text{const}$, on the other side by a temperature rise $\sqrt{t}$ the heat flux $\dot{q}|_{x=0} = \text{const}$. If the heat flux is suddenly repeatedly altered, the temperature distribution is given by

$$\vartheta - \vartheta_{O,0} = \frac{2}{b}\sum_{n=1}^{N} \Delta\dot{q}_n \cdot \sqrt{(N+1-n)\Delta t} \times i^1 \operatorname{erfc}\frac{x}{2\sqrt{(N+1-n)\Delta t}} \tag{34}$$

and the temperature on the surface

$$\vartheta_{O,N} - \vartheta_{O,0} = \frac{2}{\sqrt{\pi}\,b}\sum_{n=1}^{N} \Delta\dot{q}_n \cdot \sqrt{(N+1-n)\Delta t} \tag{35}$$

In the case of $\dot{q}|_{x=0} = \text{const}$, it is to be set for $n = 1$: $\Delta\dot{q}_1 = \dot{q}|_{x=0}$, and for $n \geqslant 2$: $\Delta\dot{q}_n = 0$. With $N \cdot \Delta t = t$ Eq. (33) is valid again.

4. Extension to finite solids—flat plates with two boundaries

In semi-infinite solids, the solutions of Fourier's equation are given as a function of the variable $\frac{at}{x^2}$. In finite solids, see Fig. 5, additional conditions for the heat transfer on both surfaces are to be stated. For convective heat transfer, it is usual to define the Biot number as a parameter for the case in question

$$\text{for } x = 0\text{: } Bi_0 = \alpha_0 s/\lambda, \quad \text{for } x = s\text{: } Bi_s = \alpha_s s/\lambda \tag{36}$$

The solutions of Eq. (1) appear in form of series with trigonometrical functions [1,2,8]. In the integral equations (5) with (6), these series are used instead of the error functions.

4.1. Boundary conditions of the 1. kind on the one side and of the 3. kind on the other side

Up to now there is assumed a sudden jump of the temperature $\Delta\vartheta_M$ on the side $x = 0$, immediately followed from the surface temperature $\Delta\vartheta_O$, i.e., $Bi_0 = \infty$, boundary condition of the 1. kind. On the other side $x = s$ a boundary condition of the 3. kind is given by $Bi_s = \alpha_s s/\lambda$. In this case, the solution for the temperature distribution gets for $Bi_0 = \infty$

$$\vartheta - \vartheta_{O,0} = \Delta\vartheta_O\left[\frac{1 + Bi_s(1 - x/s)}{1 + Bi_s} - \sum_{m=1}^{\infty}\frac{2(\mu_m^2 + Bi_s^2)\sin(\mu_m x/s)}{\mu_m(Bi_s + Bi_s^2 + \mu_m^2)}e^{-\mu_m^2 \cdot at/s^2}\right] \tag{37}$$

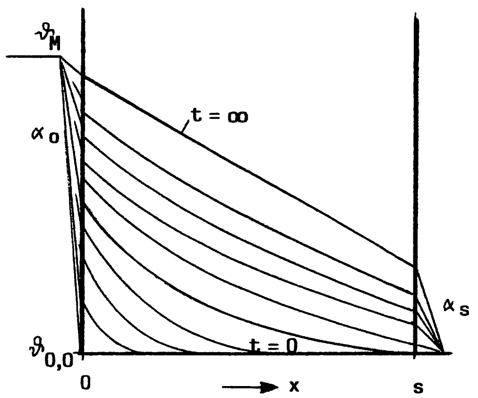

Fig. 5. Temperature distribution in a finite solid with boundary conditions of the 3. kind on both sides.

with μ_m the positive roots of the transcendent equation

$$\mu_m + Bi_s \cdot \mathrm{tg}\,\mu_m = 0 \tag{38}$$

Eq. (37) describes the stationary temperature distribution for $t \to \infty$

$$\begin{aligned}\vartheta - \vartheta_{O,0} &= \Delta\vartheta_O \frac{1 + Bi_s(1 - x/s)}{1 + Bi_s}\\ &= \Delta\vartheta_O \frac{1/\alpha_s + (s-x)/\lambda}{1/\alpha_s + s/\lambda}\end{aligned} \tag{39}$$

The temperature distribution at the time $T = N \cdot \Delta t$ can be indicated for repeated jumps $\Delta\vartheta_{0,n}$ in constant time steps Δt with the aid of Duhamel's theorem again (see Eq. (15))

$$\begin{aligned}\vartheta - \vartheta_{O,0} &= \frac{1 + Bi_s(1 - x/s)}{1 + Bi_s} \cdot \sum_{n=1}^{N} \Delta\vartheta_{O,n}\\ &\quad - \sum_{n=1}^{N} \Delta\vartheta_{O,n} \cdot \sum_{m=1}^{\infty} \frac{f_m \sin(\mu_m x/s)}{\mu_m}\\ &\quad \times e^{-(N+1-n)\cdot\mu_m^2 a\Delta t/s^2}\end{aligned} \tag{40}$$

with $f_m = 2\frac{Bi_s^2 + \mu_m^2}{Bi_s + Bi_s^2 + \mu_m^2}$.

4.2. Boundary conditions of the 3. kind on both sides of the solid

There are two ways to solve the problem, see Eqs. (20)–(23) and Eqs. (25) and (27).

(1) The temperature on the surface $x = 0$ has to be calculated at every step from the following relations for a temperature jump in the medium $\Delta\vartheta_M$ and $Bi_0 = \alpha_0 s/\lambda$

$$\alpha_0(\vartheta_{M,N} - \vartheta_{O,N}) = \dot q|_{x=0} = -\lambda \frac{\partial\vartheta}{\partial x}|_{x=0} \tag{41}$$

In connection with Eq. (40), the heat flux is defined by

$$\begin{aligned}\dot q|_{x=0} &= \frac{Bi_s \cdot \lambda/s}{1 + Bi_s}(\vartheta_{O,N} - \vartheta_{O,0})\\ &\quad + \frac{\lambda}{s}\sum_{n=1}^{N} \Delta\vartheta_{O,n} \cdot \sum_{m=1}^{\infty} f_m \cdot e^{-(N+1-n)\cdot\mu_m^2 a\Delta t/s^2}\end{aligned} \tag{42}$$

and accordingly the surface temperature

$$\begin{aligned}&\vartheta_{O,N} - \vartheta_{O,0}\\ &\quad = (\vartheta_{M,N} - \vartheta_{O,0}) Bi_s \alpha_0/\alpha_s\\ &\qquad \times \left[\frac{Bi_s\alpha_0}{\alpha_s} + \frac{Bi_s}{1 + Bi_s} + \sum_{m=1}^{\infty} f_m \cdot e^{-\mu_m^2 a\Delta t/s^2}\right]^{-1}\\ &\qquad + \sum_{n=1}^{N-1} \Delta\vartheta_{O,N-n} \cdot \sum_{m=1}^{\infty} f_m \cdot e^{-\mu_m^2 a\Delta t/s^2}\\ &\qquad \times \left(1 - e^{-\mu_m^2 n a\Delta t/s^2}\right)\\ &\qquad \times \left[\frac{Bi_s\alpha_0}{\alpha_s} + \frac{Bi_s}{1 + Bi_s} + \sum_{m=1}^{\infty} f_m \cdot e^{-\mu_m^2 a\Delta t/s^2}\right]^{-1}\end{aligned} \tag{43}$$

The sum $\sum m$ converges for small steps Δt very slowly. For short times, it is therefore useful and justified to employ the results for the semi-infinite solid, see below Section 6. (Note: when transforming Eqs. (41)–(43) the following identity is used

$$\sum_{n=1}^{N} \Delta\vartheta_{O,n} \cdot f(N+1-n) = \sum_{n=0}^{N-1} \Delta\vartheta_{O,N-n} \cdot f(n+1) \tag{44}$$

by means of which it is possible to eleminate the unknown temperature $\vartheta_{O,N}$ from the sum with $n = 0$.)

(2) The unknown surface temperature can be eliminated in accordance with the above mentioned Eq. (25). The solution for asymmetrical boundary conditions of the 3. kind—$x = 0$: $\alpha_0(\vartheta_M - \vartheta_O) = \dot q|_{x=0}$ and $x = s$: $\alpha_s(\vartheta_{x=s} - \vartheta_{O,0}) = \dot q|_{x=s}$—therefore is as follows [3]

$$\begin{aligned}&\frac{\vartheta - \vartheta_{O,0}}{\vartheta_M - \vartheta_{O,0}}\\ &\quad = k\left(\frac{s-x}{\lambda} + \frac{1}{\alpha_s}\right)\\ &\qquad - 2\sum_{m=1}^{\infty}\left\{\sin\mu_m Bi_0\left[\mu_m \cos\frac{s-x}{s}\mu_m\right.\right.\\ &\qquad\qquad \left.\left. + Bi_s \sin\mu_m \frac{s-x}{s}\right] e^{-\mu_m^2 at/s^2}\right\}\\ &\qquad \times \left\{\mu_m\left[2\mu_m \sin^2\mu_m\right.\right.\\ &\qquad\qquad \left.\left. + (Bi_0 + Bi_s)(\mu_m - \sin\mu_m \cos\mu_m)\right]\right\}^{-1}\end{aligned} \tag{45}$$

with the transcendent equation for μ_m

$$\mathrm{tg}\,\mu_m = \frac{\mu_m(Bi_0 + Bi_s)}{\mu_m^2 - Bi_0 \cdot Bi_s} \tag{46}$$

The first term in Eq. (45) represents the stationary temperature distribution for $t \to \infty$, with k the overall heat transfer coefficient $\frac{1}{k} = \frac{1}{\alpha_0} + \frac{s}{\lambda} + \frac{1}{\alpha_s}$.

The heat flux in the solid follows by differentiation of Eq. (45)

$$\begin{aligned}\dot q &= (\vartheta_M - \vartheta_{O,0})\left[k + 2\alpha_0 \sum_{m=1}^{\infty}\left[\sin\mu_m\left(\mu_m \sin\mu_m \frac{s-x}{s}\right.\right.\right.\\ &\qquad \left.\left. - Bi_s \cos\mu_m \frac{s-x}{s}\right) e^{-\mu_m^2 at/s^2}\right]\\ &\quad \times \left[2\mu_m \sin^2\mu_m\right.\\ &\qquad \left.\left. + (Bi_0 + Bi_s)(\mu_m - \sin\mu_m \cos\mu_m)\right]^{-1}\right]\end{aligned} \tag{47}$$

and for $x = 0$

$$\begin{aligned}\dot q|_{x=0} &= (\vartheta_M - \vartheta_{O,0})\\ &\quad \times \left(k + 2\alpha_0 \sum_{m=1}^{\infty} f(\mu_m, Bi_0, Bi_s) e^{-\mu_m^2 at/s^2}\right)\end{aligned} \tag{48}$$

with $f(\mu_m, Bi_0, Bi_s) = \frac{\sin\mu_m[\mu_m \sin\mu_m - Bi_s \cos\mu_m]}{2\mu_m \sin^2\mu_m + (Bi_0 + Bi_s)(\mu_m - \sin\mu_m \cos\mu_m)}$

The following is valid for repeated jumps with Duhamel's theorem

$$\dot{q}|_{x=0} = k(\vartheta_{M,N} - \vartheta_{O,0}) + 2\alpha_0 \sum_{n=1}^{N} \Delta\vartheta_{M,n} \sum_{m=1}^{\infty} f(\mu_m, Bi_0, Bi_s) \times e^{-(N+1-n)\mu_m^2 a\Delta t/s^2} \tag{49}$$

or

$$\dot{q}|_{x=0} = k(\vartheta_{M,N} - \vartheta_{O,0}) + 2\alpha_0 \sum_{n=0}^{N-1} \Delta\vartheta_{M,N-n} \sum_{m=1}^{\infty} f(\mu_m, Bi_0, Bi_s) \times e^{-(n+1)\mu_m^2 a\Delta t/s^2} \tag{50}$$

5. Breaking off the series after n_A steps, estimating the error

Small steps in the time are needed if the temperature $\Delta\vartheta_M(t)$ must be reproduced with high accuracy or if the alterations follow in a long interval. In these cases, the number of steps N can grow very high and therefore the question is whether the series can be broken off after n_A steps and whether the residual terms of the serie can be replaced by an estimation.

As it is seen in Eqs. (27), (42) or (48), the first alterations in $\Delta\vartheta_M$ hold a decreasing effect with growing numbers N of steps, i.e., the temperature distribution is being influenced by the last steps strongest. The series in the equations named above can be written in a generalized form

$$\vartheta_{M,N} - \vartheta_{O,N} = \sum_{n=0}^{N-1} \Delta_{N-n} \cdot F(n+1) \tag{51}$$

e.g., Eq. (27):

$$\vartheta_{M,N} - \vartheta_{O,N} = \sum_{n=0}^{N-1} \Delta\vartheta_{M,N-n} \cdot e^{z'^2} \cdot \operatorname{erfc}(z') \tag{52}$$

or Eq. (35)

$$\vartheta_{O,N} - \vartheta_{O,0} = \frac{2}{\sqrt{\pi}\, b} \sum_{n=0}^{N-1} \Delta\dot{q}_{N-n}\sqrt{(n+1)\Delta t} \tag{53}$$

or Eq. (48)

$$\dot{q}|_{x=0} = 2\alpha_0 \sum_{n=0}^{N-1} \Delta\vartheta_{M,N-n} \cdot f \cdot e^{-\nu(n+1)} \tag{54}$$

These sums may be divided into a correctly calculated sum and a rest R, see Fig. 6:

$$\sum_{n=0}^{N-1} \Delta_{N-n} \cdot F(n+1) = \sum_{n=0}^{n_A} \Delta_{N-n} \cdot F(n+1) + \sum_{n_A+1}^{N-1} \Delta_{N-n} \cdot F(n+1) \tag{55}$$

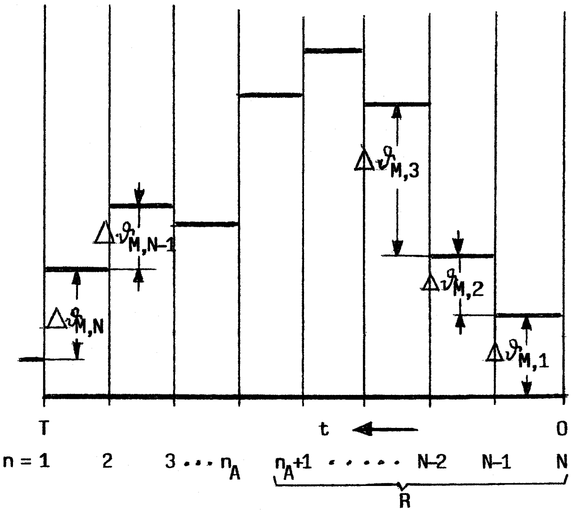

Fig. 6. Illustration of a series breaking off.

The remainder term $R = \sum_{n_A+1}^{N-1} \Delta_{N-n} \cdot F(n+1)$ has to be approximated by the sum of the jumps $\sum_{n_A+1}^{N-1} = \Delta\vartheta_{M,N-n}$ (or $\sum_{n_A+1}^{N-1} \Delta\dot{q}_{N-n}$) multiplied with a mean value $\overline{F}(n+1)$ for the number of steps $n_A + 1$ to $N - 1$:

$$R = \overline{F}(n+1) \sum_{n_A+1}^{N-1} \Delta_{N-n} \tag{56}$$

It is obvious that this mean value can be replaced by an integration instead of the sum in the same range

$$\overline{F}(n+1) = \frac{1}{(N-1)-(n_A+1)} \int_{n_A+1}^{N-1} F(n+1)\,\mathrm{d}n \tag{57}$$

The integration with the sum of Eqs. (52)–(54) leads to

$$\overline{F} = \frac{2b}{\alpha_0\sqrt{\pi}\sqrt{\Delta t}} \frac{\sqrt{N} - \sqrt{n_A+2}}{(N-1)-(n_A+1)} \tag{58}$$

respectively

$$\overline{F} = \sum \frac{2\sqrt{\Delta t}}{b\sqrt{\pi}} \cdot \frac{2}{3} \cdot \frac{N^{3/2} - (n_A+2)^{3/2}}{(N-1)-(n_A+1)} \tag{59}$$

respectively

$$\overline{F} = \frac{e^{-\nu N}}{\nu N} \cdot \frac{e^{1-(n_A+2)/N} - 1}{1-(n_A+2)/N} \tag{60}$$

(Note: the approximate equation is used in the integration of Eq. (58)

$$e^{z'^2} \cdot \operatorname{erfc}(z') \approx \frac{1}{z' \cdot \sqrt{\pi}} \tag{61}$$

for $z' = \frac{\alpha_0}{b}\sqrt{(n+1)\Delta t} > 5$. This can be done, because breaking off the sums is only possible following a great number of steps n_A.)

If the remainder R consists of only one term, $N-1=n_A+1$, Eqs. (58)–(60) result in $\overline{F}=0/0$. The limiting values are

$$R=\frac{b}{\alpha_0\sqrt{\pi}\sqrt{\Delta t}}\cdot\frac{1}{\sqrt{N}}\cdot(\vartheta_{M,N-n_A-1}-\vartheta_{M,0}) \tag{62}$$

respectively

$$R=\sum\dot{q}\cdot\sqrt{N} \tag{63}$$

respectively

$$R=f\cdot\frac{e^{-\nu\cdot N}}{\nu\cdot N}(\vartheta_{M,N-n_A-1}-\vartheta_{M,0}) \tag{64}$$

Eqs. (58) and (62) for $\overline{F}$ and R, as well as (60) and (64) converge with growing N, but not Eqs. (59) and (63) for the heat flux. Therefore, the breaking off the sums of the heat flux is not allowed. Eqs. (59) and (63) are irrelevant.

It can be recognized in the above equations that the implementation of the correctly calculated series in the remainder term R occurs continously, which is a sign that the rest term has been calculated in a suitable way.

6. Approximate solutions

The complete description of the temperature distribution from the first step to steady (or quasi-steady) temperatures requires long series with slow convergency, especially in the beginning. Several approximate solutions are known to handle this difficulty [2,6,8]. Thereby, the complete solution is divided into two parts: one part, in which the equations of the semi-infinite solid are valid at the beginning of the process, and a second part, in which the steady temperatures will be obtained. In this second part, only the first term of the series $\sum m$ with μ_1 is necessary to describe the temperature process in the time.

The example in Figs. 7 and 8 shows the heat flux $\dot{q}|_{x=0}$ for a flat plate with asymmetric boundary conditions of the 3. kind by a sudden jump $\Delta\vartheta_M$ in the fluid medium and an initial temperature $\vartheta_{O,0}\neq f(x)$, see above Section 4.2.

In Figs. 7 and 8 the heat flux $\dot{q}|_{x=0}=f(t)$, respectively, $=f(at/s^2)$ is related to the stationary heat flux $\dot{q}=k(\vartheta_M-\vartheta_{O,0})$:

(1) the complete solution after Eq. (48),
(2) the solution alone with the first term μ_1 ($\mu_1=f(Bi_0, Bi_s)$, see Fig. 9,
(3) the solution for the semi-infinite solid after Eq. (26).

These figures show:

(1) $\frac{at}{s^2}<0.3$: Eq.(26) and Eq. (48) are identical.
(2) $\frac{at}{s^2}>0.3$: solution with the first term μ_1 is identical to the complete solution.
(3) $\frac{at}{s^2}>1.6$: the steady-state has been obtained, $\dot{q}|_{x=0}/k(\vartheta_M-\vartheta_{O,0})=1$.

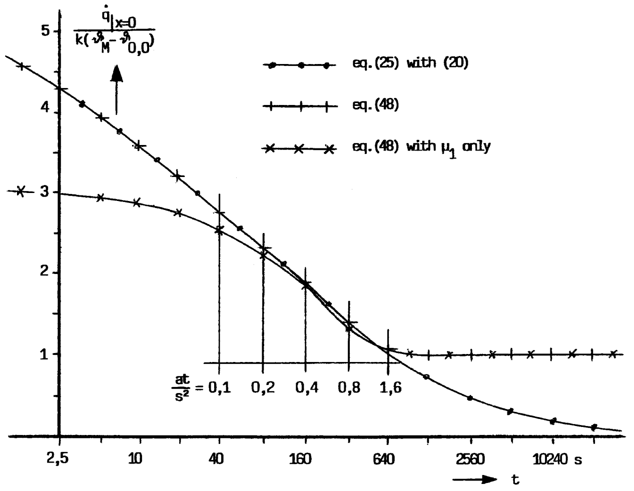

Fig. 7. Dimensionless heat flux $\dot{q}|_{x=0}$ for short times after Eq. (25) with Eq. (20), after Eq. (48) for the complete solution and after Eq. (48) for the solution with μ_1 only. The parameters are: $\alpha_0=40$ W·m^{-2}·K^{-1}, $\alpha_s=20$ W·m^{-2}·K^{-1}, $\lambda=0.4$ W·m^{-1}·K^{-1}, $c=1000$ J·kg^{-1}·K^{-1}, $\rho=400$ kg·m^{-3}, $b=400$ W·s$^{1/2}$·m^{-2}·K^{-1}, $a=10^{-6}$ m^2·s^{-1}, $k=8$ W·m^{-2}·K^{-1}, $\mu_1=1.5094$.

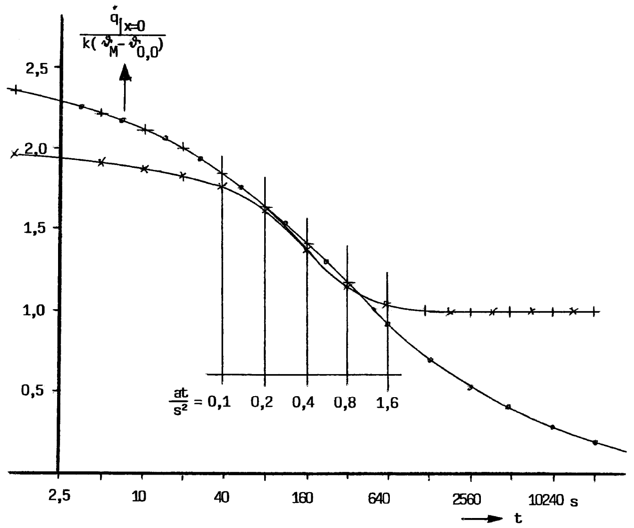

Fig. 8. Like Fig. 7, but permuted heat transfer coefficients: $\alpha_0=20$ W·m^{-2}·k^{-1}, $\alpha_s=40$ W·m^{-2}·K^{-1}.

These statements can be generalized, because the same functional dependency of the parameters is given:

$$z^2=\frac{\alpha^2 t}{\lambda c\rho}\sim\frac{\mu_1^2 at}{s^2}\quad\text{with}\quad\mu\sim\frac{\alpha s}{\lambda}.$$

Therefore, two possibilities exist for an approximate calculation of heat flux in the above problem:

(1)

$$\frac{at}{s^2}<0.3:\quad\frac{\dot{q}|_{x=0}}{k(\vartheta_M-\vartheta_{O,0})}=\frac{\alpha_0}{k}e^{z^2}\operatorname{erfc}(z) \tag{65}$$

$$\frac{at}{s^2}>0.3:\quad\frac{\dot{q}|_{x=0}}{k(\vartheta_M-\vartheta_{O,0})}=1+2\cdot\frac{\alpha_0}{k}f(\mu_1)\cdot e^{-\mu_1^2 at/s^2} \tag{66}$$

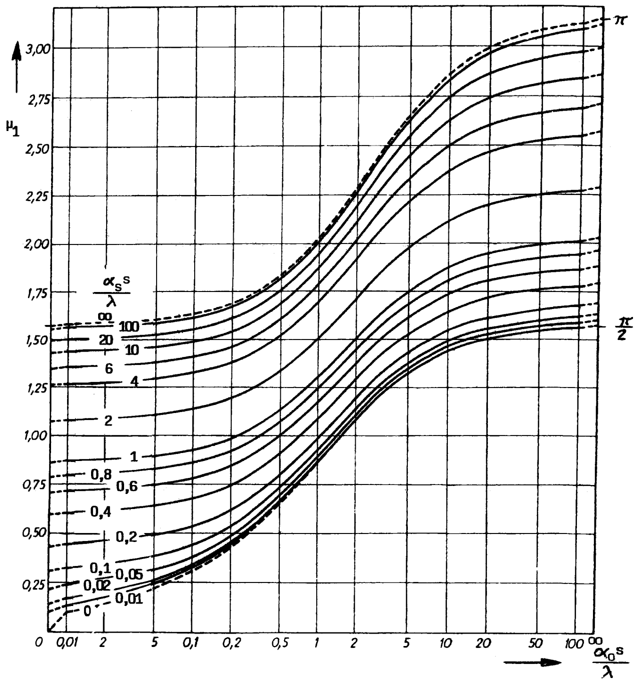

Fig. 9. The first root μ_1 of Eq. (46), from [3].

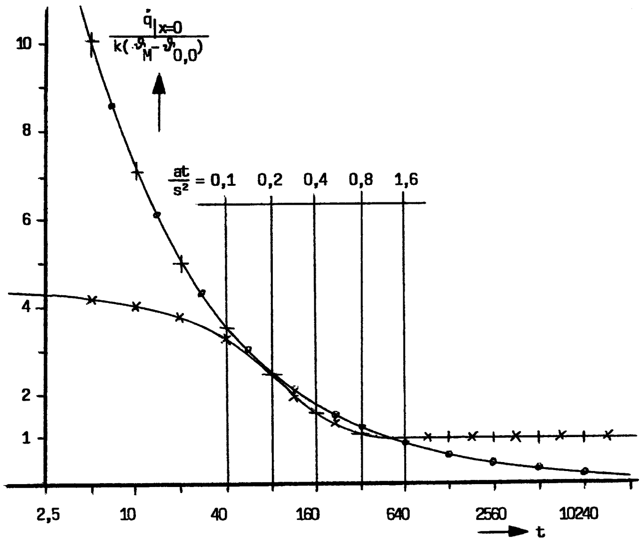

Fig. 10. Like Fig. 7, but $\alpha_0 = \infty$, $k = 10\ \mathrm{W \cdot m^{-2} \cdot K^{-1}}$, $\mu_1 = 2.0288$.

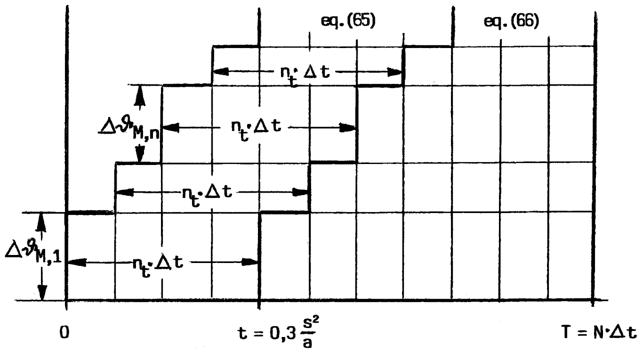

Fig. 11. Illustration to Eqs. (65) and (66).

(2)

$$\frac{\alpha_0}{k} e^{z^2} \operatorname{erfc}(z) > 1: \quad \frac{\dot{q}|_{x=0}}{k(\vartheta_M - \vartheta_{O,0})} = \frac{\alpha_0}{k} e^{z^2} \operatorname{erfc}(z) \qquad (67)$$

$$\frac{\alpha_0}{k} e^{z^2} \operatorname{erfc}(z) \leqslant 1: \quad \frac{\dot{q}|_{x=0}}{k(\vartheta_M - \vartheta_{O,0})} = 1 \qquad (68)$$

with $z = \frac{\alpha_0}{b}\sqrt{(N+1-n)\Delta t}$ and k the overall heat transfer coefficient.

The second approximation results in some bigger deviations—up to 10%—which may be admissible considering the uncertainty of other assumptions, e.g., the heat transfer coefficients.

The approximations by Eqs. (65)–(68) should be proved for a boundary condition of the first kind, i.e., $\alpha_0 = \infty$, too, see Fig. 10. This confirms the proposed approximations for the entire range of heat transfer coefficients to a flat plate.

Eqs. (65)–(68) are valid for just a single jump $\Delta\vartheta_M$. For repeated jumps, these equations can be extended using the Duhamel's theorem as shown above. Thereby it is to notice, that with every jump on the side $x = 0$, a new temperature front in the solid will be produced, which reaches the other side $x = s$ in the dimensionless time $at/s^2 = 0.3$.

When applying the approximation equations (65) and (66), Eq. (26) has to be used for a calculation if the step number $n_t < 0.3 s^2/(a\Delta t)$:

$$\frac{\dot{q}|_{x=0}}{k(\vartheta_M - \vartheta_{O,0})} = \frac{\alpha_0}{k} \sum_{n=1}^{N} \frac{\Delta\vartheta_{M,n}}{\vartheta_{M,N} - \vartheta_{O,0}} e^{z^2 (N+1-n)\Delta t} \operatorname{erfc}(z) \qquad (69)$$

and Eq. (40) with μ_1 has to be used if $n_t > 0.3 s^2/(a\Delta t)$:

$$\frac{\dot{q}|_{x=0}}{k(\vartheta_M - \vartheta_{O,0})} = 1 + 2\frac{\alpha_0}{k} \sum_{n=1}^{N} \frac{\Delta\vartheta_{M,n}}{\vartheta_{M,N} - \vartheta_{O,0}} \times f\left(\mu_1, \frac{\alpha_0 s}{\lambda}, \frac{\alpha_s s}{\lambda}\right) \cdot e^{-(N+1-n)a\mu_1^2 \Delta t/s^2} \qquad (70)$$

The number of the steps n_t is the same for all jumps $\Delta\vartheta_{M,n}$. Eq. (70) is valid for the steps $n > n_t$ with the sum $(n_t + 1)$ to N. This procedure is illustrated in Fig. 11.

When applying the approximation equations (67) and (68), every jump has to be calculated with Eq. (26) until the steady state value

$$\frac{\dot{q}|_{x=0}}{k(\vartheta_{M,N} - \vartheta_{O,0})} = \frac{\Delta\vartheta_{M,N}}{\vartheta_{M,N} - \vartheta_{O,0}} \qquad (71)$$

has been reached. When finally, all jumps $\Delta\vartheta_{M,n}$ have reached the steady state, the following equation is valid

$$\frac{\dot{q}|_{x=0}}{k(\vartheta_{M,N} - \vartheta_{O,0})} = \frac{\sum_{n=1}^{N} \Delta\vartheta_{M,n}}{\vartheta_{M,N} - \vartheta_{O,0}} = 1 \qquad (72)$$

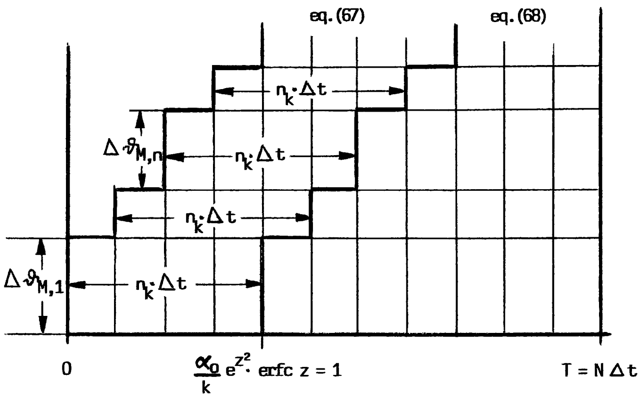

Fig. 12. Illustration to Eqs. (67) and (68).

The number of steps n_k is the same for every jump, as demonstrated in Fig. 12.

7. Summary

The theorem of Duhamel makes it possible to calculate the effects of repeated alterations in the boundary conditions—in the fluid or in the surface temperatures or in the heat transfer coefficients—for the temperature distribution and the heat flux in a solid. The alterations in the boundary conditions can be stochastical and will be adapted to an imposed or forced process by jumps with steps in the time. This procedure results in mathematical series, which will be extended by every new alteration for further steps.

The application for semi-infinite solids is possible with the aid of error function. For solids with two boundaries, e.g., flat plates, solutions are known with Fourier functions, which by Duhamel's theorem result in double series. A sufficient accuracy for technical applications will be reached by an approximate solution for the semi-infinite solid and for longer times from an approximate Fourier function (first term of the series). This process requires a change from one mode to the other during the calculations, but this can be done numerically. Based on the statements mentioned above, Duhamel's theorem is an universal tool to calculate consecutive alterations of boundary conditions and their effects on the temperature distribution and heat flux. This has been demonstrated by some examples.

Appendix A. Testing the theorem of Duhamel for a temperature oscillation by a periodic function

A further test for the extended Duhamel's theorem will be demonstrated in the application for a temperature oscillation in the medium by a sine or cosine function.

$$\vartheta_M - \vartheta_m = A\sin(\omega t) \tag{73}$$

or

$$\vartheta_M - \vartheta_m = A\cos(\omega t) \tag{74}$$

with A the amplitude of the oscillation, ϑ_m the constant mean temperature, $\omega = 2\pi/t_0$, t_0 the period. The temperature field for the stationary state ($t \to \infty$) is described for the boundary conditions of the first kind by known equations, corresponding to Eq. (73):

$$\vartheta(x,t) - \vartheta_m = Ae^{-x\sqrt{\pi/at_0}} \cdot \sin\left(\omega t - x\sqrt{\frac{\pi}{at_0}}\right) \tag{75}$$

and to Eq. (74):

$$\vartheta(x,t) - \vartheta_m = Ae^{-x\sqrt{\pi/at_0}} \cdot \cos\left(\omega t - x\sqrt{\frac{\pi}{at_0}}\right) \tag{76}$$

For the heat flux at $x = 0$, it follows with $\dot{q}|_{x=0} = -\lambda\frac{\partial\vartheta}{\partial x}$:

$$\dot{q}|_{x=0} = Ab\sqrt{\omega}\sin\left(\omega t + \frac{\pi}{4}\right) \tag{77}$$

and

$$\dot{q}|_{x=0} = Ab\sqrt{\omega}\cos\left(\omega t + \frac{\pi}{4}\right) \tag{78}$$

The preceding equations are valid, as stated above, for the stationary state after the transient disturbances at the beginning of the oscillations have disappeared. The effects of the disturbances are different for an oscillation of a sine function and of a cosine function. For a sine function, Eq. (73), the temperature rise in the beginning is nearly linear and the heat flux can be calculated with Eq. (11) and $\Delta\vartheta_0 = A\omega t\cos(\omega t)$, written in a dimentionless form

$$\frac{\dot{q}|_{x=0}}{Ab\sqrt{\omega}} = \frac{2}{\sqrt{\pi}}\sqrt{\omega t}\cos(\omega t) \tag{79}$$

with $\omega t < 0.1$. For a cosine function, Eq. (74), a jump in the surface temperature is forced for $t = 0$, $\frac{\partial\vartheta}{\partial t}|_{t=0} = \infty$. Therefore the heat flux at $t = 0$ is infinite. For small values of ωt it is valid (Eq. (17), $N = 1$):

$$\frac{\dot{q}|_{x=0}}{Ab\sqrt{\omega}} = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{\omega t}} \tag{80}$$

with $\omega t < 0.1$.

Both these cases of a harmonic oscillation can be derived from a general solution in [3] (Eqs. (11), (20)). There are two functions u^* and v^* (parts of the error function $\operatorname{erf}(z)$ with complex argument $\mathrm{f}(\sqrt{\frac{\omega t}{2}}, \frac{x}{2\sqrt{at}} + \sqrt{\frac{\omega t}{2}})$) defined. For the corrections of the heat flux by the disturbances, the differential quotients of u^* and v^* are needed:

$$\dot{q}|_{x=0} = -\lambda\frac{\partial v^*}{\partial x}\bigg|_{x=0} = -\frac{Ab\sqrt{\omega}}{2\sqrt{\omega t}} \cdot \frac{\partial v^*}{\partial(x/2\sqrt{at})}\bigg|_{x=0} \tag{81}$$

and

$$\dot{q}|_{x=0} = -\lambda\frac{\partial u^*}{\partial x}\bigg|_{x=0} = \frac{Ab\sqrt{\omega}}{2\sqrt{\omega t}} \cdot \frac{\partial u^*}{\partial(x/2\sqrt{at})}\bigg|_{x=0} \tag{82}$$

In Fig. 13 Eqs. (81) and (82) are drawn, calculated from the tabulated value in [3]. For the heat flux at $x = 0$ with

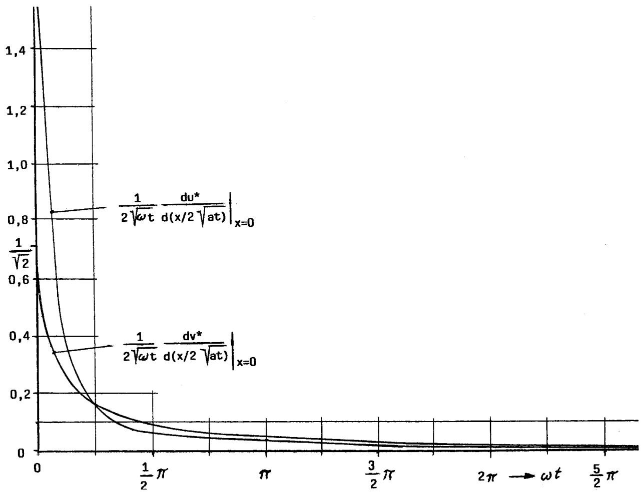

Fig. 13. Corrections of the heat flux due to disturbances.

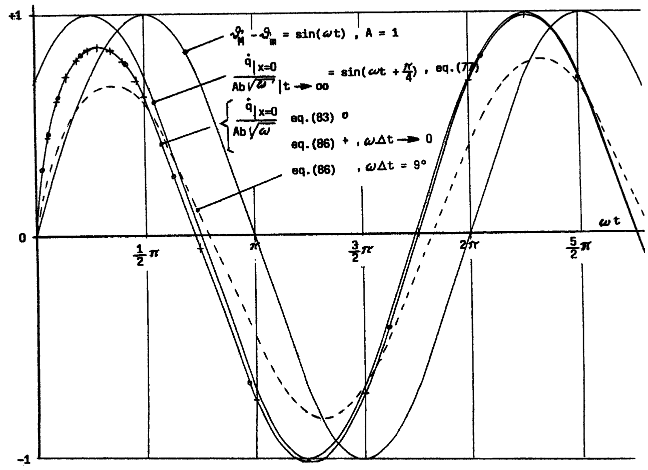

Fig. 14. Temperature oscillation by a sine function and corresponding heat flux at $x = 0$ for a boundary condition of the 1. kind in comparison of the analytical solution Eq. (83) with the Duhamel theorem Eq. (86).

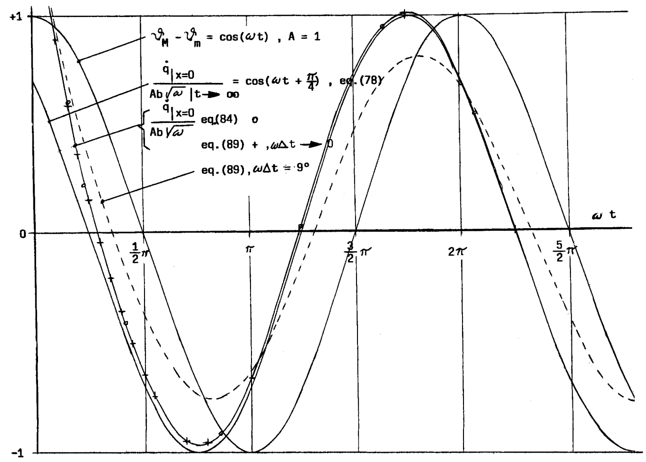

Fig. 15. Like Fig. 14, in comparison to Eqs. (84) and (89) for a temperature oscillations by a cosine function.

regard to the disturbances it follows:

$$\frac{\dot{q}|_{x=0}}{Ab\sqrt{\omega}} = \sin\left(\omega t + \frac{\pi}{4}\right) - \frac{1}{2\sqrt{\omega t}} \cdot \frac{\partial v^*}{\partial(x/2\sqrt{at})}\bigg|_{x=0} \tag{83}$$

$$\frac{\dot{q}|_{x=0}}{Ab\sqrt{\omega}} = \cos\left(\omega t + \frac{\pi}{4}\right) + \frac{1}{2\sqrt{\omega t}} \cdot \frac{\partial u^*}{\partial(x/2\sqrt{at})}\bigg|_{x=0} \tag{84}$$

It has to be proved whether the extended Duhamel theorem can be used for calculating these Eqs. (83) and (84).

The general equations of Duhamel's theorem are derived in Section 3, Eqs. (18) and (19) for a boundary condition of the 1. kind. For an oscillation of a sine function, Eq. (73), the temperature changes $\Delta\vartheta_{O,n}$ for every step Δt are given by

$$\begin{aligned}\Delta\vartheta_{O,n} &= \frac{\partial\vartheta}{\partial t}\bigg|_n \cdot \Delta t \\ &= A\omega\Delta t \cdot \cos(n\omega\Delta t), \quad n \geqslant 1\end{aligned} \tag{85}$$

and Eq. (19) for the heat flux on the surface $x = 0$ gives

$$\frac{\dot{q}|_{x=0}}{Ab\sqrt{\omega}} = \frac{\sqrt{\omega\Delta t}}{\sqrt{\pi}} \cdot \sum_{n=1}^{N} \frac{\cos(n\omega\Delta t)}{\sqrt{N+1-n}} \tag{86}$$

For an oscillation of a cosine function, Eq. (74), the first temperature change for $t = 0$ is a jump

$$\Delta\vartheta_{O,1} = A \tag{87}$$

and then the following changes obey the analogous equation to Eq. (82)

$$\Delta\vartheta_{O,n} = -A\omega\Delta t \cdot \sin(n\omega\Delta t), \quad n \geqslant 2 \tag{88}$$

In this case Eq. (19) has to be written in the form

$$\begin{aligned}\frac{\dot{q}|_{x=0}}{Ab\sqrt{\omega}} &= \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{\omega\Delta t}} \cdot \frac{1}{\sqrt{N}} \\ &\quad - \frac{\sqrt{\omega\Delta t}}{\sqrt{\pi}} \sum_{n=2}^{N} \frac{\sin(n\omega\Delta t)}{\sqrt{N+1-n}}\end{aligned} \tag{89}$$

Fig. 14 for the sine function and Fig. 15 for the cosine-function show a comparison of the analytical solutions of Eqs. (77) and (78), respectively (83) and (84) with the solutions by Duhamel's theorem Eqs. (86) and (89). It can be seen that the phase shift of $\pi/4$ sets in very rapidly and the disturbances vanish after one period of the oscillation. Most significantly, it has to be recognized that the analytical solutions and the solutions by Duhamel's theorem are almost identical. Small differences are due to numerical inaccuracies. A condition for these good agreements is to choose the time steps as small as possible, $\omega\Delta t < 0.002 = 0.1°$, even if the number of steps N will grow reciprocally to $\omega\Delta t$ (compare the dotted line and the full line for Eq. (86), respectively Eq. (89) in Figs. 14 and 15).

The examples above demonstrate in a convincing manner how the Duhamel theorem is applied for more complex temperature variations.

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